Entropy of Scalar Field in 3+1 Dimensional Reissner-Nordstrom de Sitter Black Hole Background

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Abstract

We consider the thermodynamics of minimally coupled massive scalar field in 3+1 dimensional Reissner-Nordstrom de Sitter black hole background. The brick wall model of 't Hooft is used. When Schwarzschild like coordinates are used it is found that two radial brick wall cut-off parameters are required to regularize the solution. Free energy of the scalar field is obtained through counting of states using the WKB approximation. It is found that the free energy and the entropy are divergent in both the cut-off parameters.

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I. INTRODUCTION

A black hole has a horizon beyond which no matter or information can escape. The absence of information about the region inside the horizon manifests itself in an entropy. A quantitative expression for the entropy and the laws of black hole thermodynamics were first obtained by Bekenstein [1] mainly on the basis of analogy. Since then a lot of effort has been devoted to explain this entropy on a statistical mechanics basis. A related issue is the entropy of quantum fields in black hole backgrounds. The entropy of quantum fields is obtained by various methods, e.g., by tracing over local degrees of freedom inside the horizon (geometric entropy) [2], by explicit counting of degrees of freedom of the fields propagating outside the horizon (entanglement entropy) [3,4,5] or by the Euclidean path integral [6,7]. These expressions are proportional to the area of the horizon and constitute the first quantum correction to the gravitational entropy. Divergences appear in the density of the states associated to the horizon and can be absorbed in the renormalized expression of the gravitational coupling constant [8]. To regularize these divergences 't Hooft proposed [3] that the field modes should be cut off in the vicinity of the horizon by imposing a brick wall cut-off. This method has been used to study the entropy of matter around different black hole solutions.

All recently available data from cosmological observations, including measurements of the present value of the Hubble parameter and dynamical estimates of the present energy density of the Universe, give strong sugessions that in the framework of inflationary cosmology a nonzero repulsive cosmological constant, $\Lambda \geq 0$, $(10^{16} - 10^{18}\ GeV)^2$, has to be invoked in order to explain the properties of the presently observed Universe. De Sitter space is the maximally symmetric solution of the vacuum Einstein equations with a positive cosmological constant Λ. Ginsparg and Perry [9] studied the semiclassical instabilities of de Sitter space. Due to the presence of a cosmological event horizon and its associated Hawking radiation, the de Sitter space exhibits a semiclassical instability to the nucleation of black holes, known as the Kottler [10] or Schwarzschild -de Sitter metric. This solution represents a nonrotating black hole immersed in de Sitter space. Mann and Ross [11] studied the pair creation of

electrically charged black holes in a cosmological background. There are two instantons describing the pair production of non-extreme and extreme black holes respectively. These solutions represents nonrotating electrically charged black holes immersed in de Sitter space and are called the Reissner-Nordstrom de Sitter black holes.

Hawking and Gibbons [12] studied the thermodynamical aspects and the Hawking radiation of scalar fields in the Kerr-Newman-de Sitter space. It can be shown that if we try to use the Euclidean Schwarzschild- de Sitter solution to provide thermodynamical aspects, the time periods required to avoid the conical singularities at the black hole horizon and the cosmological horizon, do not match. This is physically interpreted as indicating that the two horizons are not in thermal equilibrium and they both emit Hawking radiation at the corresponding temperatures. An observer situated somewhere between the black hole horizon and the cosmological horizon and whose world line coincides with an orbit of the static killing vector receives a thermal radiation coming from the black hole and an isotropic thermal radiation with a different temperature coming from the cosmological horizon. However, in the case of Reissner-Nordstrom de Sitter black holes it has been observed that there are two sectors of solutions depending on the values of the parameters present in the metric for which the black hole horizon and the cosmological horizon are in thermal equilibrium. These are known as the lukewarm Reissner-Nordstrom de Sitter black hole and the cold Reissner-Nordstrom de Sitter black hole solutions.

In this context it is natural to enquire about the entropy of quantum fields defined on such backgrounds. Thus we investigate the thermodynamical behavior of a massive minimally coupled real scalar field propagating on the 3+1 dimensional lukewarm Reissner-Nordstrom de Sitter black hole and the cold Reissner-Nordstrom de Sitter black hole solutions using the brick wall cut-off.

II. RENORMALIZATION OF THE GRAVITATIONAL ACTION AND ENTANGLEMENT ENTROPY

In the study of the one-loop effective action [13], we may start with the gravitational action

$$I_g = \int d^4x\sqrt{-g}[-\frac{\Lambda}{8\pi G} + \frac{R}{16\pi G}] \tag{1}$$

where Λ is the cosmological constant and G is Newton's constant. Here we have neglected the interactions which are quadratic in the curvature. The constants are bare coupling constants. In the case of Reissner-Nordstrom black hole there will be a Maxwell term and, in general, additional higher derivative interactions with the metric and gauge fields:

$$I_M = \int d^4x\sqrt{-g}[-\frac{1}{4}F_{ab}F^{ab} + \delta(F_{ab}F^{ab})^2 + \lambda R_{ab}F^{ac}F^b_c] \tag{2}$$

here δ and λ are the bare coupling constants. We also include the action for a minimally coupled neutral scalar field,

$$I_m = \int d^4x\sqrt{-g}[g^{ab}\nabla_a\psi\nabla_b\psi + m^2\psi^2] \tag{3}$$

We want to study the effective action for the metric which results when in the path integral the scalar field is integrated out. In this case , the integration is simply gausian, yielding the square root of the determinant of the propagator. The contribution to the effective gravitational action is given by,

$$W(g) = -\frac{i}{2}Trlog[-G_F(g,m^2)] \tag{4}$$

This expresssion is divergent. The divergence of this one-loop effective action and its metric dependence can be found through an adiabatic expansion for the DeWitt-Schwinger proper time representation of the propagator [14]. This leads to a representation of the scalar one-loop action as an asymptotic series [14]:

$$W(g) = -\frac{1}{32\pi^2}\int d^4x\sqrt{-g}\int_0^\infty \frac{ds}{s^3}\Sigma a_n(x)(is)^n e^{-im^2 s} \tag{5}$$

here the summation runs over all integral values of n from 0 to ∞. The adiabatic expansion coefficients $a_n(x)$ are functionals of the local geometry at x and can be constructed in terms of the metric and the curvature tensor. In the case of four dimensions the ultraviolet divergences arise as $s \to 0$ in the first three terms of the series. The corresponding adiabatic expansion coefficients are given by,

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= \tfrac{1}{6}R \\
a_2 &= \tfrac{1}{180}R^{abcd}R_{abcd} - \tfrac{1}{180}R^{ab}R_{ab} + \tfrac{1}{30}\nabla^a\nabla_a R + \tfrac{1}{72}R^2
\end{aligned}
\tag{6}
$$

The effective action may be regulated using many different methods. It is usually found [8, 13] that the term linear in the curvature,R, is quadratically divergent in the corresponding regularizing parameter and renormalizes the bare gravitational constant, *i.e*, we have,

$$\frac{1}{G_R} = \frac{1}{G} + a \tag{7}$$

here G_R is the renormalized gravitational constant and a is the quantum correction arising from the scalar field sector.

A related issue is the entropy of matter fields defined on the black hole backgrounds and in thermal equilibrium with the black hole at a temperature corresponding to that of the black hole event horizon. When one tries to find the degeneracy of the field modes, one finds an alarming divergence associated with the infinite blue shift at the event horizon. To resolve the problem 't Hooft proposed a model in which only low energy quantum fluctuations of the fields are taken into account [3]. He assumed that the usual quantum field theoretic description of the matter fields are valid up to some point close to the event horizon: $r_1 = r_+ + \epsilon,\ \epsilon \geq 0$. Correspondingly we put the following boundary condition on the scalar field wave function,

$$\psi(r,\theta,\phi,t) = 0 \ if \ r \leq r_1. \tag{8}$$

For asymptotically flat space time we also need an infrared cutoff in the form of box with a large radius L:

$$\psi(r,\theta,\phi,t) = 0 \; if \; r \geq L. \tag{9}$$

The radial degeneracy factor is obtained through a semiclassical quantization condition using the WKB approximation. The entanglement entropy is obtained through explicit counting of states. The entanglement entropy of a minimally coupled scalar field defined in asymptotically flat nonextreme Reissner-Nordstrom black hole was obtained in [5] and the ultraviolet divergent part to the leading order is given by,

$$S_m = \frac{{r_+}^2}{90h^2} \tag{10}$$

Here h is a covariant cutoff parameter and in terms of the coordinate cutoff parameter, ϵ, it is given by,

$$h = \int_{r_+}^{r_1} \frac{dr}{\sqrt{V(r)}} \tag{11}$$

The ultraviolet divergent part of the matter field entropy and the standard Bekenstein-Hawking entropy, $S_{BH} = \frac{A}{(4G)}$, where A is the area of the black hole event horizon are related in a simple way. If we add these two entropies, we have,

$$S_{BH} + S_m = \frac{A}{4}(\frac{1}{G} + \frac{1}{360\pi h^2}) \tag{12}$$

If we compare equ.(12) with equ.(7) we find that the entanglement entropy of the scalar field is in close analogy with the quantum correction of the scalar field to the bare gravitational constant arising from the one loop effective action (4). Similar observations in the case of Schwarzschild metric led Susskind and Uglum to put forward, for the case of canonical quantum gravity coupled to matter fields, the following conjecture:

The expression $S_{BH} = \frac{A}{4\pi}$ for the Bekenststein-Hawking entropy of the fields propagating outside a black hole is a general result, but the gravitational coupling arising is the renormalized gravitational coupling G_R given by equ.(7).

The brick wall model was used by G 't Hooft in a different context[3]. He wanted to explain the black hole entropy in terms of matter fields living outside the horizon and in thermal equilibrium with the black bole. If we equate S_m with the Bekenstein-Hawking

value S_{BH}, the cut-off parameter is found out to be of the order of the Planck length and it is given by $h = \sqrt{\frac{G}{90\pi}}$.

An exceptional situation arises when one considers the case of the extremal Reissner-Nordstrom black hole in asymptotically flat space time. In this case the gravitational entropy and the temperature of the event horizon vanishes [5]. The entanglement entropy of a minimally coupled massive scalar field in this case is given by [5],

$$S_m \sim \frac{8\pi^3}{135}(r_+/\beta)^3 exp(3\lambda/r_+) \tag{13}$$

where $\lambda \rightarrow \infty$ is a large cutoff parameter. The exponential divergence can be traced through the fact that the horizon is actually infinite distance away along any constant time hypersurface. If we take the standard value of the event horizon temperature, which is zero, the matter field entropy vanishes and we do not face any problem. In the Euclidean sector the horizon is infinitely far away along all directions. This means that the Euclidean solution can be identified with any temperature β and the black hole can be in equilibrium with thermal radiation at any temperature. However with β arbitrary the expression (13) is exponentially divergent and with no corresponding term from the gravitational sector, it is not obvious how to interpret S_m in terms of quantum corrections to the renormalization of the gravitational constant G.

In this context it is interesting to study the entanglement entropy of a minimally coupled scalar field in cold Reissner-Nordstrom de Sitter background where an extremal black hole is immersed in the surrounding Hawking radiation coming from the cosmological horizon with a finite temperature.

III. CALCULATION OF ENTANGLEMENT ENTROPY

The Reissner-Nordstrom de Sitter metric is given by,

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{14}$$

where

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2 \qquad (15)$$

for electrically charged solution the gauge field is,

$$F = -\frac{Q}{r^2} dt \wedge dr \qquad (16)$$

This solution has three independent parameters,the 'mass' M, charge Q, and cosmological constant Λ, all of which are positive. There are four roots of $V(r)$, which we designate by r_1, r_2, r_3, r_4 in ascending order. In the Lorentzian section, $0 \le r < \infty$. The first root is negative and has no physical significance. The second root r_2 is the inner (Cauchy) black hole horizon, $r = r_3$ is the outer (Killing) horizon, and $r = r_4$ is the cosmological (acceleration) horizon. The Killing vector $K = \frac{\partial}{\partial t}$ is uniquely defined by the conditions that it be null on both the Killing horizon and the cosmological horizons and that its magnitude should tend $(\frac{\Lambda}{3})^{1/2} r$ as r tends to infinity. For the cold Reissner-Nordstrom de Sitter metric $V(r)$ has a double root. In this case the Cauchy horizon and the Killing horizon coincide, i, e $r_2 = r_3 = \rho$ and we have,

$$V(r) = (1 - \frac{\rho}{r})^2 [1 - \frac{\Lambda}{3}(r^2 + 2\rho r + 3\rho^2)] \qquad (17)$$

the critical relationships between mass, charge, event horizon radius and the cosmological constant is,

$$M = \rho(1 - \tfrac{2}{3}\Lambda\rho^2) \qquad (18)$$

$$Q^2 = \rho^2(1 - \Lambda\rho^2)$$

For Λ positive, there is a maximum allowed radius, $\rho = \rho_m = \Lambda^{-1/2}$, at which $Q = 0$. The resulting metric is the Nariai metric and is characterized by, $M = \frac{\rho}{3}, \quad Q^2 = 0, \quad \Lambda = \rho^{-2}$, and

$$V_N(r) = -\frac{r^2}{3\rho^2}(1 - \frac{\rho}{r})^2(1 + \frac{2\rho}{r}) \qquad (19)$$

For $0 < \rho < \rho_m = \Lambda^{-\frac{1}{2}}$, there is an extra positive root b to $V(r)$, given by,

$$b = \sqrt{(3/\Lambda) - 2\rho^2} - \rho \qquad (20)$$

For the range $0 < \rho^2 < \frac{2}{\Lambda}$, the extra horizon at b is outside the cold horizon at ρ. We have,

$$M = \frac{\rho(b+\rho)^2}{b^2+2\rho b+3\rho^2}, \qquad Q^2 = \frac{b\rho^2(b+2\rho)}{b^2+2\rho b+3\rho^2}, \qquad \Lambda = \frac{3}{b^2+2\rho b+3\rho^2} \tag{21}$$

the metric function becomes,

$$V(r) = \frac{r^2}{b^2+2\rho b+3\rho^2}(1-\frac{\rho}{r})^2(\frac{b}{r}-1)(1+\frac{2\rho+b}{r}) \tag{22}$$

the Hawking temperature, which is given by the surface gravity at the cosmological horizon at b, is:

$$T_b = \frac{b}{2\pi b^2+2\rho b+3\rho^2}(1-\frac{\rho}{b})^2(1+\frac{\rho}{b}) \tag{23}$$

for small Λ $\;(\Lambda \ll \rho^{-2}, or b \gg \rho)$, b is the outer edge of the cosmological horizon. In the Euclidean sector the double root in $V(r)$ implies that the proper distance from any point between ρ and b to $r=\rho$ along spacelike directions is infinite. In this case, we may obtain a regular instanton by identifying the imaginary time τ periodically with period $T_b = \frac{k_b}{2\pi}$, where k_b is the surface gravity of the cosmological event horizon. It is obvious that the cold Reissner-Nordstrom de Sitter solution represents an extremal black hole immersed in the thermal bath of the Hawking radiation coming from the cosmological horizon. Consequently, in the corresponding expression to equ.(13), we can take the temperature to be that of the cosmological horizon.

The wave equation for a minimally coupled scalar field in a curved background is

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi) - m^2\psi = 0. \tag{24}$$

The region of physical interest is $\rho < r < b$. In this region the space time is static and allows a global time-like killing vector. Since there is no explicit time dependence in the foliation used, we may construct stationary state solutions. We take the stationary state solutions to be of the form

$$\psi = Ke^{iEt}e^{iN\phi}P(\theta)R(r) \tag{25}$$

where N is an integer, K a normalization constant, E a real parameter and represents the energy of the scalar field. We have, using the separation of variables,

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta P) + [l(l+1) - \frac{N^2}{\sin^2\theta}]P = 0, \tag{26}$$

The solution of this equation is given by the standard associated Legendre polynomials. For the radial part, we have,

$$\frac{E^2}{V(r)}R(r) + \frac{1}{r^2}\partial_r[r^2V(r)\partial_r R(r)] - (\frac{l(l+1)}{r^2} + m^2)R(r) = 0 \tag{27}$$

We want to calculate the entropy of the scalar field. For this purpose we use the WKB approximation to the radial differential equation to obtain the radial degeneracy factor associated with the brick wall boundary condition. In this case the blue-shift factor diverges at both the black hole event horizon and the cosmological event horizon. Following 't Hooft's procedure, we introduce two brick wall cut-offs, one near the black hole event horizon and the other near the cosmological horizon by setting

$$\begin{aligned} R(r) &= 0 \quad \text{for} \quad r \leq \rho + \epsilon_1 \\ &= 0 \quad \text{for} \quad r \geq b - \epsilon_2 \end{aligned} \tag{28}$$

where $\epsilon_1, \epsilon_2 \ll \rho$.

In the WKB approximation we have $R(r) = \alpha(r)e^{iS(r)}$, where $\alpha(r)$ is a slowly varying amplitude and $S(r)$ is a rapidly varying phase factor. When substituted in equ.(11), this gives us the following value for the r-dependent wave number

$$k^2(r,l,E) = \frac{1}{V(r)^2}[E^2 - V(r)(\frac{l(l+1)}{r^2} + m^2)] \tag{29}$$

The radial degeneracy factor is obtained through the following semiclassical quantization condition:

$$\pi n_r(l,E) = \int_{\rho+\epsilon_1}^{b-\epsilon_2} k(r,l,E)dr \tag{30}$$

where it is implicitly assumed that the integration is carried over those values of l for which $k(r,l,E)$ is real.

The total number of wave solutions with energy not exceeding E, $g(E)$, is then given by,

$$g(E) = \int (2l+1)dl\ \pi n_r(l,E) \tag{31}$$

Every energy level determined as above may be occupied with any nonnegative number of quanta. The free energy at an inverse temperature β is:

$$\begin{aligned}\pi\beta F &= \int dg(E) ln(1-e^{\beta E}) \\ &= -\beta\int_0^\infty \frac{dE}{(e^{\beta E}-1)}\int_{\rho+\epsilon_1}^{b-\epsilon_2}\frac{dr}{V(r)}\int(2l+1)dl\sqrt{E^2-V(r)[\frac{m^2+l(l+1)}{r}]}\end{aligned} \tag{32}$$

When we perform the l integration over the values for which the square root is positive, we have,

$$\beta F = -\frac{2\beta}{3\pi}\int_0^\infty \frac{dE}{(e^{\beta E}-1)}\int_{\rho+\epsilon_1}^{b-\epsilon_2} dr\frac{r^2}{V(r)^2}[E^2-V(r)m^2] \tag{33}$$

To the leading order this gives us the following contribution to the free energy of the scalar field:

$$F(\beta) = -\frac{2\pi^3}{15\beta^4}\frac{\rho^6(b^2+2b\rho+3\rho^2)^2}{(b-\rho)^2(b+3\rho)^2{\epsilon_1}^3} - \frac{\pi^3}{90\beta^4}\frac{b^6(b^2+2b\rho+3\rho^2)^2}{(b-\rho)^2(b+\rho)^2\epsilon_2} \tag{34}$$

The radial brick wall cut-off parameters as introduced in equation (28) are non-covariant. We replace these by the covariant cut-off parameter h as introduced in equation (11). For the case of the cosmological horizon, $r=b$, we have,

$$h_2 = \sqrt{\frac{b^2(b^2+2b\rho+3\rho^2)^2}{2(b-\rho)^2(b+\rho)}}\int_{b-\epsilon_2}^{b}\frac{dr}{\sqrt{b-r}} \tag{35}$$

this gives,

$$\epsilon_2 = \frac{(b-\rho)^2(b+\rho)}{2b^2(b^2+2b\rho+3\rho^2)}{h_2}^2 \tag{36}$$

For the the case of the black hole Killing horizon, this issue is non-trivial due to presence of the double root in the metric function at $r=\rho$. In this case we have,

$$h_1 = \sqrt{\frac{(b^2+2b\rho+3\rho^2)}{(b-\rho)(b+3\rho)}}\int_\rho^{\rho+\epsilon_1}\frac{rdr}{r-\rho} \tag{37}$$

this is undefined at the horizon. Following [5], we set the cut-off in terms of the proper radial variable, defined by,

$$ds = \frac{dr^2}{V(r)^2} \tag{38}$$

The horizon is at $s = -\infty$. Thus the cut-off is at a large negative distance $s = -\lambda$, where

$$-\lambda = p\rho[1 + ln(\epsilon_1/\rho)] + p\epsilon_1, \quad p = \sqrt{\frac{(b^2 + 2b\rho + 3\rho^2)}{(b-\rho)(b+3\rho)}} \tag{39}$$

this gives,

$$\epsilon_1 \sim \rho exp[-(\frac{\lambda}{p\rho})] \tag{40}$$

The entropy is given by

$$S = \beta^2 \frac{dF}{d\beta} \tag{41}$$

when ϵ_1, ϵ_2 are replaced by the covariant cut-off parameters and for the value of temperature as given in expression (23), this leads to,

$$S \sim \frac{\rho^3}{15b^3} exp(\frac{3\lambda}{p\rho}) + \frac{b^2}{90h^2} \tag{42}$$

here $\lambda \to \infty$ and $h \to 0$.

In the case of the Euclidean lukewarm solution the conical singularities at both the outer Killing horizon and the cosmological event horizon are removed by identifying the imaginary time coordinate with the same period. In this case the black hole horizon is at thermal equilibrium with the cosmological horizon. This leads to the following condition,

$$V(a) = V(b); \quad V'(a) = V'(b) \tag{43}$$

this leads to,

$$V_{lu}(r) = \frac{(r-a)(b-r)}{r^2(a+b)^2}[r^2 + r(a+b) - ab] \tag{44}$$

where a and b are the outer black hole horizon and the cosmolological horizon. We have the following relations:

$$M = \frac{ab}{(a+b)}, \qquad Q^2 = [\frac{ab}{(a+b)}]^2, \qquad \Lambda = \frac{3}{(a+b)} \tag{45}$$

The common temperature at a and b is,

$$T = \frac{(b-a)}{2\pi(a+b)^2} \tag{46}$$

The entanglement entropy of a minimally coupled scalar field, to the leading order, is given by,

$$S = \frac{a^2}{1440{h_1}^2} + \frac{b^2}{1440{h_2}^2} \tag{47}$$

where $h_1, h_2 \to 0$, are the two covariant cut-off parameters associated with the black hole horizon and the cosmological horizon respectively.

IV. DISCUSSION

The Bekenstein-Hawking entropy for the cold Reissner-Nordstrom de Sitter black hole was obtained by Mann and Ross [11]. It is given by,

$$S_{BH} = \pi b^2 \ = A/4 \tag{48}$$

The total entropy is then given by:

$$\begin{aligned} S = &\qquad S_{BH} + S_m \\ &= \pi b^2 \left(\tfrac{1}{G} + \tfrac{1}{90\pi h_2^2}\right) + \tfrac{\rho^3}{15b^3} exp(3\lambda) \\ &= \qquad \tfrac{A}{4G_R} + \tfrac{\rho^3}{15b^3} exp(3\lambda) \end{aligned} \tag{49}$$

where G_R is the renormalized gravitational constant. As mentioned earlier cold Reissner-Nordstrom de Sitter black hole represents an extremal black hole immersed in the de Sitter space. Here, in the corresponding expression to equ.(13) we have a natural choice for the value of temperature, the temperature of the cosmological horizon. In this case we find that the gravitational constant is renormalized through the ultraviolet divergent part of the scalar field entropy associated with the cosmological horizon. The divergent contribution associated with the black hole horizon, apart from the exponential factor, behaves like the infrared divergent part arising in the asymptotically flat Reissner-Nordstrom black hole. The black hole horizon plays the role of a boundary at the internal infinity.

In the case of the lukewarm solution the Bekenstein-Hawking entropy is given by,

$$S_{BH} = \pi a^2 + \pi b^2 \quad = (A_B + A_C)/4 \tag{50}$$

here A_B and A_C are the black hole horizon and the cosmological horizon surface area. The total entropy is given by,

$$\begin{aligned} S = & \quad S_{BH} + S_m \\ & = \pi a^2 (\tfrac{1}{G} + \tfrac{1}{1440\pi h_1^2}) + \pi b^2 (\tfrac{1}{G} + \tfrac{1}{1440\pi h_1^2}) \end{aligned} \tag{51}$$

Now, without any lose of generality, we can identify h_1 *and* h_2. In that case we have,

$$S = (A_B + A_C)/4G_R \tag{52}$$

$i.e$, the ultraviolet divergences of the scalar field entanglement entropy associated with the black hole event horizon and the cosmological event horizon gives quantum corrections to the renormalized gravitational constant G_R.

As discussed earlier we can fix the value of the covariant cut-off parameter $'h'$ by equating the entanglement entropy to the gravitational entropy. In the case of the cold Reissner-Nordstrom de Sitter black hole, we can equate the ultraviolet divergent part associated with the cosmological horizon of the scalar field entanglement entropy to the corresponding gravitational entropy. The cut-off parameter "h" in this case is given by $h = \sqrt{\frac{G}{90\pi}}$. In the case of the lukewarm solution the cut-off is given by $h = \sqrt{\frac{G}{1440\pi}}$. In the units chosen both the cut-off are of the order of the Planck length although their values are different. This is expected because the cold Reissner-Nordstrom de Sitter black hole and the lukewarm Reissner-Nordstrom de Sitter black hole are geometrically two different spacetime.

We conclude with a brief comment on the Schwarzschild-de Sitter metric. In this case the black hole horizon radius is very small compared to that of the cosmological horizon radius. Consequently, the black hole Hawking radiation is at higher temperature than that of the Hawking radiation coming from the cosmological horizon and it is not obvious how to define the temperature of the matter field between the two horizon. Similar situation also arises in the case Reissner-Nordstrom de Sitter black hole when $M \neq Q$. The entanglement entropy of a scalar field was carried out in [15] in the limit $r_4 \gg r_3$. In this case the temperature at which the black hole radiates is much larger than that of the cosmological horizon, and, the temperature of the matter field was taken to be that of the outer Killing horizon. However, in

the semicassical appriximation, $G\Lambda \ll 1$, the Schwarzschild-de Sitter black hole evaporates away on a time scale much shorter than that required for another nucleation [9] and we can preferably define a test matter field at the temperature of the cosmological horizon. This issue is currently under study.

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